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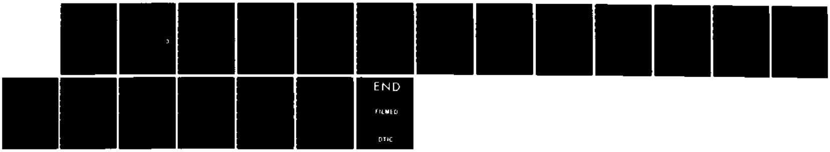

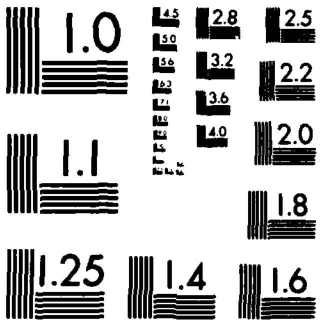

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WITH INCREASING FAILURE RATE

by Mark Brown
City College, CUNY

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Summary. Inequalities are obtained for IFR (increasing failure rate) distributions. These include bounds on the renewal function for a renewal process with IFR interarrival time, and bounds on the quality of exponential approximation to IFR distributions.

1. Introduction.

Suppose that F has an IFR (increasing failure rate) distribution with mean μ, second moment μ_2, stationary renewal distribution $G(t) = \mu^{-1} \int_0^t \bar{F}(x)dx$, and $\rho = 1-(\mu_2/2\mu^2)$. Consider a renewal process with interarrival time distribution F, and define M(t) to be the expected number of renewals in [0,t], including a renewal at time zero.

Marshall and Proschan (1972) showed that for F NBUE (new better than used in expectation, a weaker property than IFR):

$$(1.1) \qquad M(t) \le \frac{t}{\mu} + 1 .$$

In section 2 it is shown that for F IFR:

$$(1.3) \qquad M(t) \ge \frac{t}{\mu} + \frac{\sigma^2}{\mu^2} .$$

Thus for F IFR, (1.1) and (1.2) combine to give the two sided bound:

$$(1.3) \qquad \frac{t}{\mu} + \frac{\sigma^2}{\mu^2} \le M(t) \le \frac{t}{\mu} + 1 .$$

For F non-lattice with finite second moment $M(t)-t\mu^{-1}-(\mu_2/2\mu^2)$ converges to 0 as $t \to \infty$ (Feller (1971) p. 366), and thus $t\mu^{-1}+(\mu_2/2\mu^2)$ serves as an asymptotic linear approximation to M(t). Defining $L(t) = M(t)-t\mu^{-1}-(\mu_2/2\mu^2)$, the error of approximation at t, it follows from (1.3) that for F IFR:

(1.4) $$-\rho \le L(t) \le \rho$$

(1.5) $$\sup_t |L(t)| = \rho = L(0)$$

Thus ρ equals the exact sup norm error for the asymptotic linear approximation.

The parameter ρ was suggested by Keilson (1975) as a measure of departure of a distribution from an exponential distribution with the same mean. Inequality (1.5) demonstrates that in the IFR case ρ measures a characteristic of the departure of the renewal process with distribution F from that of a Poisson process with the same mean interarrival time.

Results are also obtained for the approximate exponentiality of IFR distributions with small ρ. For probability distributions F_1, F_2 on $[0,\infty)$ define $D(F_1,F_2) = \sup|F_1(t)-F_2(t)|$, and $D^*(F_1,F_2) = \sup|F_1(B)-F_2(B)|$, the sup taken overall Borel subsets of $[0,\infty)$. Define aE to be an exponential distribution with mean a. In section 3 the following inequalities are derived:

(1.6) $$D(F,\mu E) \le 2\rho$$

(1.7) $$D^*(F,G) \le 2\rho$$

(1.8) $$D^*(G,\mu E) \le \rho$$

(1.9) $$D(G,\mu_G E) \le \rho .$$

Thus for F IFR with small ρ, F and G are approximately equal and approximately exponential.

Brown and Ge (1984) showed that for F IFRA (increasing failure rate on the average) the best bound for $D(F,\mu E)$ of the form $c\rho^{\alpha}$ has $\alpha = \frac{1}{2}$ and $1 \le c \le \frac{4\sqrt{6}}{\pi}$. Thus (1.5) cannot be extended from IFR to IFRA. However, (1.5) does extend to the class of absolutely continuous distributions which are simultaneously IFRA and DMRL (decreasing mean residual life). I don't know whether or not (1.5) holds for the class of DMRL distributions.

Finally it is shown that for F IFR:

$$(1.10) \qquad \overline{F}(t) \le e^{-t\mu^{-1}+2\rho} \quad \text{for} \quad t \ge 0 .$$

This result combines with an inequality of Barlow and Proschan ((1975) p. 113) to yield:

$$(1.11) \qquad e^{-t\mu^{-1}} \le \overline{F}(t) \le e^{-t\mu^{-1}+2\rho} \quad \text{for} \quad 0 \le t \le \mu .$$

The methodology of this paper overlaps with that of Brown (1980) and (1983). However, I have found the IFR class to be more difficult to penetrate than DFR for the properties of interest. In some cases no close analogue of the DFR result holds, in others the IFR analogue is weaker.

Increasing failure rate distributions on $[0,\infty)$ are absolutely continuous except perhaps for an atom at the right hand endpoint of the support. The atom leads to uninteresting technicalities and in this paper we ignore them by assuming the IFR distributions to be absolutely continuous. All the above results are for atomless IFR distributions. However, I believe that they hold in the general case. The method of proof would be to replace the atom at the right hand endpoint, b, by a uniform distribution on $[b-\varepsilon,b]$ with the same mass as the atom, and let $\varepsilon \to 0$. The resulting distributions are absolutely continuous IFR distributions which converge to the original. Then continuity arguments are needed to show that the corresponding functionals converge. This line of argument is not pursued here.

Increasing failure rate distribuitons have been widely studied. Some notable references are Barlow, Marshall and Proschan (1963), Barlow and Marshall (1964a,b), Barlow and Proschan (1964), and Barlow (1965). A lucid discussion of the subject can be found in Barlow and Proschan (1975). Bounds on the renewal function have been investigated in the general case by Lorden (1970), Stone (1972) and Daley (1976), (1978) and for reliability classes by Brown (1980) and Marshall and Proschan (1972).

2. Renewal Function Inequalities.

A distribution on $[0,\infty)$ is defined to be IFR (Barlow and Proschan (1975) p. 54) if the residual life is stochastically decreasing in t i.e. $\overline{F}(t+s)/\overline{F}(t)$ is decreasing in t for each $s > 0$. IFR distributions can have support $[0,\infty)$ in which case they are absolutely continuous, or support $[a,b]$ with $0 \leq a \leq b < \infty$ in which case they are absolutely continuous except perhaps for an atom at b (Barlow and Proschan (1975) p. 77). As mentioned in the introduction, we will assume without further mention that the IFR distributions have no atom, and thus are absolutely continuous. The IFR property is equivalent to $H(t) = -\ln\bar{F}(t)$ convex, and to $h(t) = H'(t) = f(t)/\bar{F}(t)$ increasing, where h is the failure rate function (Barlow and Proschan (1975) p. 54).

We will refer at times to classes defined by weaker aging properties than IFR. The class IFRA (increasing failure rate on the average) is characterized by H starshaped, i.e. $H(t)/t$ increasing, a weaker property than H convex; DMRL (decreasing mean residual life) distributions have $E(X-t|X>t)$ decreasing, a weaker property than $X-t|X>t$ stochastically decreasing; NBU (new better than used) distributions have $X-t|X>t$ stochastically smaller than X for all $t \geq 0$, and NBUE (new better than used in expectation) distributions satisfy $E(X-t|X>t) \leq EX$ for all $t \geq 0$. All the above classes are discussed in Barlow and Proschan (1975).

Lemma 2.1 below rephrases and simplifies Lemma 3.3 of Brown (1980):

<u>Lemma 2.1</u>. Assume that F_1 and F_2 are probability distributions on the real line with $\bar{F}_1(t)/\bar{F}_2(t)$ increasing in t. Then there exists (X_1,X_2) with $X_1 \sim F_1$, $X_2 \sim F_2$, $X_1 \geq X_2$ a.s. and:

$$(2.2) \quad D^*(F_1,F_2) = \sup_B |F_1(B)-F_2(B)| \leq \Pr(X_1 > X_2) \leq 1 - \int \frac{\bar{F}_2(t^-)}{\bar{F}_1(t^-)} dF_1(t) .$$

<u>Proof</u>. Define $\bar{F}_Z(t) = \bar{F}_2(t)/\bar{F}_1(t)$ and note that $\bar{F}_Z(-\infty) = 1$, $\bar{F}_Z$ is decreasing and right continuous and $0 \leq \bar{F}_Z(t) \leq 1$ for all t; $\bar{F}_Z$ is thus the survival function of a possibly improper random variable. Construct X_1 and Z independent with $X_1 \sim F_1$, $Z \sim F_Z$ and define

$X_2 = \min(X_1, Z)$, noting that $\Pr(X_2 > t) = \overline{F}_2(t)$, thus $X_2 \sim F_2$. For any Borel set B:

$$|F_1(B)-F_2(B)| = |\Pr(X_1 \varepsilon B, X_1 \neq X_2) - \Pr(X_2 \varepsilon B, X_1 \neq X_2)|$$

$$\leq \max(\Pr(X_1 \varepsilon B, X_1 \neq X_2), \Pr(X_2 \varepsilon B, X_1 \neq X_2))$$

$$\leq \Pr(X_1 > X_2) = \Pr(Z < X_1) = 1 - \int \frac{\overline{F}_2(t^-)}{\overline{F}_1(t^-)} dF_1(t) .$$

<u>Theorem 2.3</u>. Suppose that F is IFR and that G is the stationary renewal distribution corresponding to F. Then there exists (X_1, X_2) with $X_1 \sim F$, $X_2 \sim G$, $X_1 \geq X_2$ a.s. and:

$$D^*(F,G) \leq \Pr(X_1 > X_2) \leq 1 - \frac{\sigma^2}{\mu^2} = 2\rho \tag{2.4}$$

where $\mu = E_F X$, $\sigma^2 = \mathrm{Var}_F X$, $\rho = 1 - \frac{\mu_2}{2\mu^2}$, $\mu_2 = E_F X^2$.

<u>Proof</u>. F IFR implies $E(X-t|X > t) = \frac{\mu\overline{G}(t)}{\overline{F}(t)}$ decreasing. Thus Lemma 2.1 is applicable with $F_1 = F$ and $F_2 = G$. This gives:

$$D^*(F,G) \leq 1 - \int \overline{G}(t)h(t)dt \tag{2.5}$$

where h is the failure rate function of F.

Integration by parts in (2.5) produces:

$$D^*(F,G) \leq 1 - \frac{I}{\mu} \quad \text{where} \quad I = \int \overline{F}(t)H(t)dt \tag{2.6}$$

and $H(t) = -\ln\overline{F}(t) = \int_0^t h(x)dx$.

Noting that xH differentiates to $xh+H$, we have:

$$(2.7) \qquad EXH(X) = E\int_0^X (xh+H)dx = \int (xh+H)\overline{F}dx = \mu + I \ .$$

Next, noting that $EH(X) = 1$, define probability measures P and Q by:

$$P(A) = \int_A HdF; \ Q(A) = \int_A \frac{x}{\mu} dF \ .$$

The ratio of the Radon-Nikodym derivitives of P and Q with respect to F is $\mu H(x)/x$ which is increasing since F is IFR and H convex and thus starshaped. Therefore P is bigger than Q under the partial ordering of monotone likelihood ratio (Lehmann (1959) p. 74) and thus has a bigger mean. Thus

$$(2.8) \qquad EXH(X) = E_P X \geq E_Q X = \mu_2/\mu \ .$$

From (2.7) and (2.8) we conclude:

$$(2.9) \qquad I \geq \sigma^2/\mu \ .$$

The result now follows from (2.6) and (2.9). □

We now construct $\{N(t), N^*(t), t \geq 0\}$ where $\{N(t), t \geq 0\}$, is distributed as an ordinary renewal process with interarrival time distribution F(IFR) and $\{N^*(t), t \geq 0\}$ distributed as the stationary renewal process corresponding to N. The construction is similar to Brown (1980) p. 230. The process N starts with a renewal at time 0. Its next renewal occurs at $X_1 \sim F$ while the first renewal for the process N^* occurs at $Y_1 \sim G$. By Theorem 2.3 we can construct (X_1, Y_1) with $X_1 \geq Y_1$. If $X_1 = Y_1$ then we construct all future renewal epochs identical for the two processes. If $X_1 > Y_1$, then at time Y_1 process N^* has its next interarrival time, $T_2 - Y_1 \sim F$ while process N has a forward recurrence time at Y_1 distributed as $X_1 - Y_1 | X_1 > Y_1$. Since F is IFR, for any t:

$$\text{(2.10)} \qquad \frac{\overline{F}(x)}{\overline{F}(t+x)/\overline{F}(t)} \text{ is increasing in } x .$$

It follows from (2.10) and Theorem 2.3 that we can construct $(T_2 - Y_1, X_1 - Y_1)$ and thus (T_2, X_1) with $X_1 \leq T_2$ a.s.. If $X_1 = T_2$ we make all future renewal epochs identical for N and N^*, otherwise we continue the construction. We wind up with processes N and N^*, N having renewals at $0, X_1, S_2, S_3, \ldots$ N^* at $Y_1, T_2, T_3 \ldots$ with renewal epochs alternating between N and N^* until a random event epoch where both processes have a common renewal (called the coupling time) at which time they share all future event epochs. Thus a typical realization may look like:

$$0 < Y_1 = T_1 < X_1 = S_1 < T_2 < S_2 < T_3 < S_3 = T_4 ,$$

in which case $S_k = T_{k+1}$ for $k \geq 3$.

Note that under the above construction $N(t) - N^*(t)$ starts at 1, alternates between 1 and 0, and either identically equals 0 or 1 from the coupling time to ∞. In the atomless case it is easy to show that with probability one N and N^* eventually do have a common renewal epoch. With the above construction we can now derive the renewal results.

Define $M(t)$ to be the expected number of renewals, including a renewal at zero, for a renewal process with IFR interarrival time distribution F.

<u>Theorem 2.11</u>. For F IFR, the following inequalities hold:

$$\frac{t}{\mu} + \frac{\sigma^2}{\mu^2} \leq M(t) \leq \frac{t}{\mu} + 1 \tag{2.12}$$

$$-\rho \leq L(t) = M(t) - t\mu^{-1} - (\mu_2/2\mu^2) \leq \rho \tag{2.13}$$

$$\sup|L(t)| = \rho = L(0) . \tag{2.14}$$

<u>Proof</u>. By our observation that under the above construction $N(t)-N^*(t)$ can only equal 0 or 1:

$$M(t) - \frac{t}{\mu} = \Pr(N(t)-N^*(t)=1) .$$

If $X_1 = Y_1$ then $N(t)-N^*(t) \equiv 1$, for all t. From Theorem (2.3), $\Pr(X_1=Y_1)$ under the construction is at least σ^2/μ^2. Thus

$$M(t) - \frac{t}{\mu} \geq \Pr(X_1=Y_1) \geq \frac{\sigma^2}{\mu^2} . \tag{2.15}$$

Thus (2.12) follows from (2.15) and the Marshall-Proschan inequality for NBUE distributions mentioned in the introduction. Inequality (2.13) follows from (2.12) by subtracting $t\mu^{-1}+(\mu_2/2\mu^2)$ from all 3 sides of the inequality. Finally (2.14) follows from (2.13) and the observation that $L(0) = \rho$. $\square$

Analogues of the results of Brown (1980) for renewal theory for DFR interarrival times do not hold in the IFR case. An example of Berman (1978) p. 429 shows that F IFR does not imply an increasing renewal density function, nor $M(t)-\mu^{-1}t$ decreasing, nor the expected forward recurrence time decreasing. The identity (2.14) holds for F IFR (increasing mean residual life) with $F(0) = 0$, as follows from Brown (1980), Theorem 2.

3. Exponential Approximations.

Theorem (2.3) bounds the distance between F, an IFR distribution, and G its stationary renewal distribution. This bound is the key to obtaining approximate exponentiality for F under small ρ.

Theorem 3.1. Suppose that F is IFR with mean μ, second moment μ_2, $\rho = 1-(\mu_2/2\mu^2)$, and $G(t) = \mu^{-1}\int_0^t \overline{F}(x)dx$. Then:

(3.2) $$D^*(F,G) \leq 2\rho$$

(3.3) $$D^*(G,\mu E) \leq \rho$$

(3.4) $$D(F,\mu E) \leq 2\rho$$

(3.5) $$D(G,\mu_G E) \leq \rho \, .$$

Proof. Bound (3.2) is the conclusion of Theorem 2.3. Inequality (3.3) follows from Brown (1983) remark 4.14, as F is IFR and therefore NBUE. Since G is stochastically smaller than both F and $\mu\mathcal{E}$, it follows that $D(F,\mu\mathcal{E}) \leq \max(D(F,G),D(G,\mu\mathcal{E})) \leq 2\rho$ by (3.2) and (3.3), thus (3.4) is true. Finally $\mu\mathcal{E}$ is stochastically larger than both G and $\mu_G\mathcal{E}$, and $D(\mu\mathcal{E},\mu_G\mathcal{E}) \leq \rho$ by Lemma 2.1, thus $D(G,\mu_G\mathcal{E}) \leq \max(D(G,\mu\mathcal{E}),D(\mu_G\mathcal{E},\mu\mathcal{E})) \leq \rho$. □

Inequality (3.2) is sharp in that 2ρ is the best possible upper bound for $D^*(F,G)$ of the form $c\rho^\alpha$. This can be seen by letting F be a one point distribution at 1 in which case $\rho = \frac{1}{2}$ and $D^*(F,G) = 1$. F can be approximated by a uniform distribution on $[1-\varepsilon,1]$, which is IFR and absolutely continuous; as $\varepsilon \to 0$, $\rho \to \frac{1}{2}$ and $D^*(F,G) \to 1$. In this example, $D(F,\mu\mathcal{E}) = 1-e^{-1}$, thus the maximum potential improvement of inequality (3.4) is from 2ρ to $2(1-e^{-1})\rho \approx 1.26\rho$. In Brown and Ge (1984) I reported that 2ρ was the best possible bound of the form $c\rho^\alpha$, but the example I based that on contained a numerical error. The sharpness of (3.4) is still an open question.

Finally we prove the following inequality for F IFR:

$$\overline{F}(t) \leq e^{-t\mu^{-1}+2\rho} \quad . \tag{3.5}$$

This combines with a bound of Barlow and Proschan (1975). p. 113 to give the following two-sided bound:

$$e^{-t\mu^{-1}} \leq \overline{F}(t) \leq e^{-t\mu^{-1}+2\rho} \quad \text{for} \quad 0 \leq t < \mu \; . \tag{3.6}$$

The proof of (3.5) now follows. Consider N a renewal process with interarrival time F, and R a non-homogeneous Poisson process with intensity function h (the failure rate of F). For F NBU (new better than used) $N(0,t]$ is trivially stochastically smaller than $R(0,t]$ for all t. Thus:

$$M(t)-1 = EN(0,t] \le ER(0,t] = H(t) \ . \tag{3.7}$$

As a consequence of (3.7) and (2.12):

$$\overline{F}(t) = e^{-H(t)} \le e^{-(M(t)-1)} \le e^{-\frac{t}{\mu}+2\rho} \ . \tag{3.8}$$

4. <u>Comments and Additions</u>.

(4.1) A key ingredient of this paper is inequality (2.9). Following the same approach the following correlation inequality can be derived. Let $K(x)$ be starshaped and X a non-negative random variable. Then:

$$\rho(X,K(X)) \ge \frac{\sigma_X/\mu_X}{\sigma_{K(X)}/\mu_{K(X)}} \ . \tag{4.2}$$

Thus the correlation between X and $K(X)$ is bounded below by the ratio of coefficients of variation.

(4.3) Theorems 2.3 and 3.1 hold for absolutely continuous distributions which are simultaneously IFRA and DMRL, a slightly more general class than IFR. The DMRL condition appears essential, but perhaps the results hold without assuming that H is starshaped (and thus F is IFRA). What would be needed to extend the results to the class DMRL is a proof of (2.8) assuming only that F is DMRL.

(4.4) Define $Z(t)$ to be the forward recurrence time at t for a renewal process with IFR interarrival time distribution. An immediate consequence of (2.12), using Wald's identity is:

$$\frac{\sigma^2}{\mu} \leq EZ(t) \leq \mu. \tag{4.5}$$

References

Berman, M. (1978). Regenerative multivariate point processes. Advances in Appl. Probability, 10, 411-430.

Barlow, R.E. (1965). Bounds on integrals with applications to reliability problems. Ann. Math. Statist., 36, 565-574.

Barlow, R.E. and Marshall, A.W. (1964a). Bounds for distributions with monotone hazard rate, I. Ann. Math. Statist., 35, 1237-1257.

Barlow, R.E. and Marshall, A.W. (1964b). Bounds for distributions with monotone hazard rate, II. Ann. Math. Statist., 35, 1258-1274.

Barlow, R.E., Marshall, A.W., and Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. Ann. Math. Statist., 34, 375-389.

Barlow, R.E. and Proschan, F. (1964). Comparison of replacement policies, and renewal theory implications. Ann. Math. Statist., 35, 577-589.

Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rhinehart and Winston, New York.

Brown, M. (1980). Bounds, inequalities, and monotonicity properties for some specialized renewal processes. Ann. Probab., 8, 227-240.

Brown, M. (1983). Approximating IMRL distributions by exponential distributions, with applications to first passage times. Ann. Probab., 11, 419-427.

Brown, M. and Ge, G. (1984). Exponential approximations for two classes of aging distributions. Ann. Probab., 12, 869-875.

Daley, D.J. (1976). Another upper bound for the renewal function. Ann. Probab., 4, 109-114.

Daley, D.J. (1978). Upper bounds for the renewal function via Fourier methods. Ann. Probab., 876-884.

Feller, W. (1971). An Introduction to Probability Theory and its Applications, Volume II, 2nd Edition. John Wiley, New York.

Keilson, J. (1979). Markov Chain Models: Rarity and Exponentiality. Springer, New York.

Lehmann, E.L. (1959). Testing Statistical Hypotheses. John Wiley, New York.

Lordon, G. (1970). On excess over the boundary. Ann. Math. Statist., 41, 520-527.

Marshall, A.W. and Proschan, F. (1972). Classes of distributions applicable in replacement with renewal theory implications. In Proc. Sixth Berkeley Symp. Math. Statist. Probability I, edited by L. LeCam, J. Neyman and E.L. Scott, 395-415, University of California Press.

Stone, C.J. (1972). An upper bound for the renewal function. Ann. Math. Statist., 43, 2050-2053.

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